On Revision of Partially Specified Convex Probabilistic Belief Bases

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Abstract. We propose a method for an agent to revise its incomplete probabilistic beliefs when a new piece of propositional information is observed. In this work, an agent’s beliefs are represented by a set of probabilistic formulae – a belief base. The method involves determining a representative set of ‘boundary’ probability distributions consistent with the current belief base, revising each of these probability distributions and then translating the revised information into a new belief base. We use a version of Lewis Imaging as the revision operation. The correctness of the approach is proved. An analysis of the approach is done against six rationality postulates. The expressivity of the belief bases under consideration are rather restricted, but has some applications. We also discuss methods of belief base revision employing the notion of optimum entropy, and point out some of the benefits and difficulties in those methods. Both the boundary distribution method and the optimum entropy methods are reasonable, yet yield different results.

1 INTRODUCTION

Suppose an agent represents its probabilistic knowledge with a set of statements; every statement says something about the probability of some features the agent is aware of. Ideally, the agent would want to have enough information to, at least, identify one probability distribution over all the situations (worlds) it deems possible. However, if the agent could not gather sufficient data or if it was not told or given sufficient information, it would not be able to pinpoint exactly one probability distribution. An agent with this sort of ignorance, can be thought of as having beliefs compatible with a set of distributions. Now, this agent might need to revise its beliefs when new (non-probabilistic) information is received, even though the agent’s beliefs do not characterize a particular probability distribution over its current possible worlds.

Several researchers argue that using a single probability distribution requires the agent to make unrealistically precise uncertainty distinctions [9, 27, 29]. “One widely-used approach to dealing with this has been to consider sets of probability measures as a way of modeling uncertainty,” [9]. However, simply applying standard probabilistic conditioning to each of the measures/distributions in the set individually and then combining the results is either not recommended because it produces unsatisfactory results [9, 27] or because it is not computable due to the set being infinite. The framework presented in this paper proposes two ways to go from one ‘probabilistically incomplete’ belief base to another when new information is acquired.

Both belief revision methods presented, essentially follow this process: From the original belief base, determine a relatively small set of belief states/probability distributions which are ‘compatible’ with the belief base and in a sense, representative of the belief base. (We shall use the terms belief state and probability distribution interchangeably). Then revise only the belief states in this representative set. Finally, induce a new, revised belief base from the revised representative set.

We shall present two approaches to determine the representative set of belief states from the current belief base: (i) The approach we focus on involves finding belief states which, in a sense, are at the boundaries of the constraints implied by the belief base. These ‘boundary belief states’ can be thought of as drawing the outline of the convex space of beliefs. This outline is then revised to form a new outline shape, which can be translated into a new belief base. (ii) As a possible alternative approach, the representative set is a single belief state which can be imagined to be at the center of the outline of the first approach. This ‘central’ belief state is found by determining the one in the space of beliefs which is least biased or most entropic in terms of information theory [11, 4].

For approach (i) – where the canonical set is the set of boundary belief states – we shall prove that the revised canonical set characterizes the set of all belief states which would have resulted from revising all (including interior) belief states compatible with the original belief base.

The next section provides the relevant background theory and notation. Section 3 presents a generalized imaging method for revising probabilistic belief states. Then we describe the application of generalized imaging in our main contribution; revising boundary belief states instead of all belief states. The subsequent section explains two other approaches of revising our belief bases, based on optimum entropy. The first method finds a single representative belief state through maximum entropy inference and the second method revises boundary belief states using minimum cross-entropy inference. All three methods can be considered motivated methods, yet yield different results. Then, in Section 6, we shall list six traditional rationality postulates, and check how well the main approach fares against them. The related work is discussed in Section 7. We end with a section on future possible directions of research and some concluding remarks.

2 PRELIMINARIES

We shall work with classical propositional logic. Let \( P \) be the finite set of \( n \) atomic propositional variables (atoms, for short). Formally,
a world is a unique assignment of truth values to all the atoms in \( \mathcal{P} \). There are thus \( 2^n \) conceivable worlds. An agent may consider some non-empty subset \( W' \) of the conceivable worlds; \( W' \) is called the possible worlds. Often, in the exposition of this paper, a world will be referred to by its truth vector. For instance, if the vocabulary is placed in order \( (q, r) \) and \( w_3 \models \neg q \land r \), then \( w_3 \) may be referred to as \( 011 \). Let \( L \) be all propositional formulae which can be formed from \( \mathcal{P} \) and the logical connectives \( \wedge \) and \( \neg \), with \( \top \) abbreviating tautology and \( \bot \) abbreviating contradiction. Let \( \beta \) be a sentence in \( L \). \([\beta]\) denotes the set of \( \beta \)-worlds, that is, the elements of \( W \) satisfying \( \beta \). The worlds satisfying all sentences in a set of sentences \( K \) are denoted by \([K]\). We define the probabilistic language \( L_{\text{prob}} = \{ \{ \alpha \} \models x \mid \alpha \in L, x \in \{ \leq, =, \geq \}, x \in [0, 1] \} \). We propose a belief base (BB) to be a consistent (logically satisfiable) subset of \( L_{\text{prob}} \). A BB specifies an agent’s knowledge. The basic semantic element of an agent’s beliefs is a probability distribution or a belief state

\[
\mathbf{b} = \{(w_1, p_1), (w_2, p_2), \ldots, (w_n, p_n)\},
\]

where \( p_i \) is the probability that \( w_i \) is the actual world in which the agent is. We may also use \( e \) to refer to a belief state. For parsimony, let \( b = \langle p_1, \ldots, p_n \rangle \) be the probabilities that belief state assigns to \( w_1, \ldots, w_n \) where \( (w_1, w_2, w_3, w_4) = (11, 10, 01, 00), \) and \( (w_5, w_2, w_3, w_6) = (111, 110, \ldots, 000) \). Let \( \Pi \) be the set of all belief states over \( W \).

\[
b(\alpha) \text{ abbreviates } \sum_{w \in W, w \models \alpha} b(w), \quad \text{if } b(\alpha) > 0.
\]

If \( B \) is a set of formulae, then \( b(\alpha) \) for all \( \alpha \in B \). Let \( \gamma \) be the unique assignment of truth values to all the atoms in \( \mathcal{P} \) where \( \gamma(\alpha) = 0 \) iff \( \alpha \notin B \). \( \Pi_B \) is the set of \( \pi \)-worlds closest to \( \gamma \). B is logically equivalent to \( B' \) (denoted \( B \equiv B' \)) iff for all \( \pi \in \Pi_B \), \( \pi(\gamma) \). Lewis's original solution for accommodating contradicting evidence \( \alpha \) is to move the probability of each world to its closest, equally proximate, closest worlds. Gärdenfors \([7]\) calls one generalization of Lewis’s imaging, \textit{general imaging}. Our method is also a generalization of Lewis’s imaging. We thus refer to his as Gärdenfors’ \textit{general imaging} and to our method as \textit{generalized imaging} to distinguish them. It should be noted that all three these imaging methods are general revision methods and can be used in place of Bayesian conditioning for expansion.

**3 GENERALIZED IMAGING**

It is not yet universally agreed what revision means in a probabilistic setting. One school of thought says that probabilistic expansion is equivalent to Bayesian conditioning. This is evidenced by Bayesian conditioning (BC) being defined only when \( b(\alpha) \neq 0 \), thus making BC expansion equivalent to BC revision. In other words, one could define expansion (restricted revision) to be

\[
b(\alpha) = \{(w, p) \mid w \in W, p = b(w | \alpha), b(\alpha) \neq 0\},
\]

where \( b(w | \alpha) \) can be defined as \( b(\phi_w \land \alpha)/b(\alpha) \) and \( \phi_w \) is a sentence identifying \( w \) (i.e., a complete theory for \( w \)). To accommodate cases where \( b(\alpha) = 0 \), that is, where \( \alpha \) contradicts the agent’s current beliefs and its beliefs need to be revised in the stronger sense, we shall make use of imaging. Imaging was introduced by Lewis \([20]\) as a means of revising a probability distribution, and has been discussed in other work too [7, 6, 3, 25]. Informally, Lewis’s original solution for accommodating contradicting evidence \( \alpha \) to move the probability of each world to its closest, \( \alpha \)-world. Lewis made the strong assumption that every world has a unique closest \( \alpha \)-world. More general versions of imaging allows worlds to have several, equally proximate, closest worlds.

**Example 1** Let the vocabulary be \( \{q, r, s\} \). Let \( \alpha \) be \( (q \land r) \lor (q \land \neg r) \land s \). Suppose \( d \) is Hamming distance. Then

\[
\begin{align*}
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 111, d) = \{111\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 110, d) = \{110\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 101, d) = \{101\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 100, d) = \{110, 101\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 011, d) = \{111\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 010, d) = \{110\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 001, d) = \{101\} \\
\text{Min}(q \land r) & \lor (q \land \neg r) \land s, 000, d) = \{110, 101\}
\end{align*}
\]
Definition 2 (Gl) Then generalized imaging (denoted Gl) is defined as
\[
b \text{Gl } \alpha := \{ (w, p) \mid w \in W, p = 0 \text{ if } w \notin [\alpha]\},
\]
e else \[
p = \sum_{w' \in W} b(w')/|\text{Min}(\alpha, w', d)|.
\]
In words, b Gl \alpha is the new belief state produced by taking the generalized image of b with respect to \alpha. Notice how the probability mass of non-\alpha-worlds is shifted to their closest \alpha-worlds. If a non-\alpha-world w^∞ with probability \rho has \rho closest \alpha-worlds (equally distant), then each of these closest \alpha-worlds gets \rho/n mass from w^∞.

We define b^Gl_\alpha(w) := b \circ \alpha so that we can write b^Gl_\alpha(w), where \circ is a revision operator.

Example 2 Continuing on Example 1: Let b = \{0, 0.1, 0.2, 0.3, 0.4\}.

(q \land r) \lor (q \land \neg r \land s) is abbreviated as \alpha.

\[
b^\text{Gl}_\alpha(111) = \sum_{w' \in W} b(w')/|\text{Min}(\alpha, w', d)|
\]
= b(111)/|\text{Min}(\alpha, 111, d)| + b(011)/|\text{Min}(\alpha, 011, d)| = 0/1 + 0/1 = 0.

\[
b^\text{Gl}_\alpha(110) = \sum_{110 \in W} b(w')/|\text{Min}(\alpha, w', d)|
\]
= b(110)/|\text{Min}(\alpha, 110, d)| + b(100)/|\text{Min}(\alpha, 100, d)| + b(010)/|\text{Min}(\alpha, 010, d)| + b(000)/|\text{Min}(\alpha, 000, d)| = 0.1/1 + 0.2/2 + 0.3/1 + 0.4/2 = 0.7.

\[
b^\text{Gl}_\alpha(101) = \sum_{101 \in W} b(w')/|\text{Min}(\alpha, w', d)|
\]
= b(101)/|\text{Min}(\alpha, 101, d)| + b(100)/|\text{Min}(\alpha, 100, d)| + b(001)/|\text{Min}(\alpha, 001, d)| + b(000)/|\text{Min}(\alpha, 000, d)| = 0.1/1 + 0.2/2 + 0.1/1 + 0.4/2 = 0.3.

And b^\text{Gl}_\alpha(100) = b^\text{Gl}_\alpha(011) = b^\text{Gl}_\alpha(010) = b^\text{Gl}_\alpha(001) = b^\text{Gl}_\alpha(000) = 0.

4 REVISION VIA Gl AND BOUNDARY BELIEF STATES

The most obvious way to revise a given belief base (BB) B is to revise every individual belief state in \Pi B and then induce a new BB from the set of revised belief states. Formally, given observation \alpha, first determine a new belief state b^\alpha for every b \in \Pi B via the defined revision operation:

\[
\Pi ^\text{bb} := \{ b^\alpha \in \Pi \mid b^\alpha = b \circ \alpha, b \in \Pi \}
\]

If there is more than only a single belief state in \Pi ^\text{bb}, then \Pi B contains an infinite number of belief states. Then how can one compute \Pi ^\text{bb}? And how would one subsequently determine B^\alpha from \Pi ^\text{bb}? In the rest of this section we shall present a finite method of determining \Pi ^\text{bb}. What makes this method possible is the insight that \Pi B can be represented by a finite set of "boundary" belief states — those belief states which, in a sense, represent the limits or the convex hull of \Pi B. We shall prove that the set of revised boundary belief states defines \Pi ^\text{bb}. Inducing B^\alpha from \Pi ^\text{bb} is then relatively easy, as will be seen.

Let W^\text{perm} be every permutation on the ordering of worlds in W. For instance, if W = \{w_1, w_2, w_3, w_4\}, then W^\text{perm} = \{\{w_1, w_2, w_3, w_4\}, \{w_1, w_2, w_4, w_3\}, \{w_1, w_3, w_2, w_4\}, \ldots \{w_4, w_3, w_2, w_1\}\}. Given an ordering \Pi^\# \in W^\text{perm}, let \Pi^\#(i) be the i-th element of W^\#, for instance, \{w_4, w_3, w_2, w_1\}(2) = w_3. Suppose we are given a BB B. We now define a function which, given a permutation of worlds, returns a belief state where worlds earlier in the ordering are assigned maximal probabilities according to the boundary values enforced by B.

Definition 3 MaxASAP(B, W^\#) is the b \in \Pi B such that for i = 1, \ldots, |W|, \forall' \in \Pi B, if b' \neq \Pi B, then \sum_{j=1}^{|W|^\#}\Pi B^\Pi (j) \geq \sum_{k=1}^{|W|^\#}\Pi B^\Pi (k).

Example 3 Suppose the vocabulary is \{q, r\} and B_1 = \{q\} \geq 0.6. Then, for instance, MaxASAP(B_1, \{01, 00, 11, 10\}) = \{01.0.4, 00.0, 11.0.6, 10.0\} = \{11.0.6, 10.0, 01.0.4, 00.0\}.

Definition 4 We define the boundary belief states of BB B as the set \Pi^\text{bb} := \{ b \in \Pi B \mid W^\text{perm} \in W \text{perm}, b = \Pi^\text{bb}\}.

Example 4 Suppose the vocabulary is \{q, r\} and B_1 = \{q\} \geq 0.6. Then

\[
\Pi^\text{bb} := \{ \Pi^\text{bb}(11.1.0), (10.0.0), (01.0.0), (00.0.0), \}
\]
\{11.0.0, 10.1.0, (01.0.0), (00.0.0), \}
\{11.0.6, 10.0.0, (01.0.4), (00.0.0), \}
\{11.0.6, 10.0.0, (01.0.0), (00.4.0), \}
\{11.0.0, 10.0.6, (01.0.4), (00.0.0), \}
\{11.0.0, 10.0.6, (01.0.0), (00.4.0), \}
\{11.0.0, 10.0.6, (01.0.0), (00.4.0)\}.

Next, the revision operation is applied to every belief state in \Pi^\text{bb}.

Let \Pi^\text{bb}(\Pi \text{bb}) := \{ b' \in \Pi \mid b' = b^\alpha, b \in \Pi^\text{bb}\}.

Example 5 Suppose the vocabulary is \{q, r\} and B_1 = \{q\} \geq 0.6. Let b be (q \land \neg r) \lor (\neg q \land r). Then

\[
\Pi^\text{bb}(\Pi \text{bb}) := \{ \Pi^\text{bb}(11.0.0), (10.0.5), (01.0.5), (00.0.0), \}
\{11.0.0, (10.1.0), (01.0.0), (00.0.0), \}
\{11.0.0, (10.0.3), (01.0.7), (00.0.0), \}
\{11.0.0, (10.0.6), (01.0.4), (00.0.0), \}
\{11.0.0, (10.0.8), (01.0.2), (00.0.0)\}.
\]

(Two revision operations produce \{11.0, (10.0.5), (01.0.5), (00.0)\}.)

To induce the new BB B^\#_\alpha from \Pi^\text{bb}(\Pi \text{bb}) , the following procedure is executed. For every possible world, the procedure adds a sentence enforcing the upper (resp., lower) probability limit of the world, with respect to all the revised boundary belief states. Trivial limits are excepted.
For every \( w \in W \), \( (\phi_w) \leq \gamma \in B^w \), where \( \gamma = \max_{b \in (\Pi^B)_{\text{rev}}(\gamma)} b(w) \), except when \( \gamma = 1 \), and \( (\phi_w) \geq \gamma \in B^w \), where \( \gamma = \min_{b \in (\Pi^B)_{\text{rev}}(\gamma)} b(w) \), except when \( \gamma = 0 \).
if \( \overline{b}_x(w) = \overline{b}_X(w) \), where

\[
\overline{b}_x(w) := \max_{b_x \in B^{[0,1]}_x}\sum_{w' \in W \in \text{Min}(\alpha, w', d)} b_x(w')/|\text{Min}(\alpha, w', d)|
\]

and

\[
\overline{b}_X(w) := \max_{b_x \in B^x} \sum_{w' \in W \in \text{Min}(\alpha, w', d)} b_x(w')/|\text{Min}(\alpha, w', d)|.
\]

Note that

\[
\sum_{w' \in W \in \text{Min}(\alpha, w', d)} b(w')/|\text{Min}(\alpha, w', d)|
\]

can be written in the form

\[
\sum_{w' \in [w^1]} b(w') + \cdots + \sum_{w' \in [w^m]} b(w')
\]

Then by Observation 1, \( \overline{b}_X(w) \) is in \( B_{\text{bound}}^x \). And also by Lemma 1, the belief state in \( B_{\text{bound}} \) identified by \( \overline{b}_X(w) \) must be the one which maximizes

\[
\sum_{w' \in W \in \text{Min}(\alpha, w', d)} b_x(w')/|\text{Min}(\alpha, w', d)|,
\]

where \( b_x \in B^x \). That is, \( \overline{b}_x = \overline{b}_X \).

With a symmetrical argument, it can be shown that \( y_w = y_w \).

Informally, the following theorem says that the BB determined through the method of revising boundary belief states captures exactly the same beliefs and ignorance as the belief states in \( B^x \) which have been revised. This correspondence relies on the fact that the upper and lower probability envelopes of \( B^x \) can be induced from \( B_{\text{bound}}^x \), which is what Lemma 2 states.

**Theorem 1** Let \( (B^x)^{\text{GB}} := \{b^x \in B \mid b \in B^x\} \). Let \( B_{\text{bound}}^x \) be the BB induced from \( (B_{\text{bound}}^x)^{\text{GB}} \). Then \( B_{\text{bound}}^x = (B^x)^{\text{GB}} \).

**Proof:**

We show that \( \forall b \in B, b \in B_{\text{bound}}^x \iff b \in (B^x)^{\text{GB}} \).

\[
\iff b \in B_{\text{bound}}^x \implies \forall w \in W, y_w \leq b(w) \leq y_w \text{ (by definition of } B_{\text{bound}}^x) \],
\]

(\( \Leftarrow \)) \( B_{\text{bound}}^x \) implies \( \forall w \in W, y_w = y_w \), and \( y_w = y_w \). Hence, \( \forall w \in W, y_w \leq b(w) \leq y_w \). Therefore, \( b(w) \in (B^x)^{\text{GB}} \).

(\( \Rightarrow \)) \( b(w) \in (B^x)^{\text{GB}} \) implies \( \forall w \in W, y_w \leq b(w) \leq y_w \). Hence, by Lemma 2, \( \forall w \in W, y_w \leq b(w) \leq y_w \). Therefore, by definition of \( B_{\text{bound}}^x, b \in B_{\text{bound}}^x \).

\( \square \)

### 5 REVISION VIA OPTIMUM ENTROPY INFERENCE

Another approach to the revision of a belief base (BB) is to determine a representative of \( B^x \) (call it \( b_{\text{rev}} \)), change the representative belief state via the the defined revision operation and then induce a new BB from the revised representative belief state. Selecting a representative probability distribution from a family of such functions is not new [8, 22, e.g.]. More formally, given observation \( \alpha \), first determine \( b_{\text{rev}} \in B^x \), then compute its revision \( b_{\text{rev}}^\alpha \), and finally induce \( B^\alpha \) from \( b_{\text{rev}}^\alpha \).

We shall represent \( B^x \) (and thus \( B \)) by the single ‘least biased’ belief state, that is, the belief state in \( B^x \) with highest entropy:

**Definition 5 (Shannon Entropy)**

\[
H(b) := -\sum_{w \in W} b(w) \ln b(w),
\]

where \( b \) is a belief state.

**Definition 6 (Maximum Entropy)** Traditionally, given some set of distributions \( \Pi \), the most entropic distribution in \( \Pi \) is defined as

\[
b^\Pi := \arg\max_{b \in \Pi} H(b).
\]

Suppose \( B_2 = \{(\neg q \land \neg r) = 0.1\} \). Then the belief state \( b \in B_{B_2} \) satisfying the constraints posed by \( B_2 \) for which \( H(b) \) is maximized is \( b_{\text{rev}} = b^H = \{0.3, 0.3, 0.3, 0.1\} \).

The above distribution can be found directly by applying the principle of maximum entropy: The true belief state is estimated to be the one consistent with known constraints, but is otherwise as unbiased as possible, or “Given no other knowledge, assume that everything is as random as possible. That is, the probabilities are distributed as uniformly as possible consistent with the available information.” [24]. Obviously world 00 must be assigned probability 0.1. And the remaining 0.9 probability mass should be uniformly spread across the other three worlds.

Applying GL to \( b_{\text{rev}} \) on evidence \( \neg q \) results in \( b_{\text{rev}}^\neg q = \{0, 0, 0.6, 0.4\} \).

**Example 8** Suppose the vocabulary is \( \{q, r\} \), \( B_1 = \{(q) \geq 0.6\} \) and \( \alpha \) is \( (q \land \neg r) \lor (\neg q \land r) \). Then \( b_{\text{rev}} = \arg\max_{b \in \Pi} H(b) = \{0.3, 0.3, 0.2, 0.2\} \). Applying GL to \( b_{\text{rev}} \) on \( \alpha \) results in \( b_{\text{rev}}^\alpha = \{0, 0.61, 0.39, 0\} \). \( b_{\text{rev}}^\alpha \) can be translated into \( B_{\text{rev}}^\alpha \) as \( \{q \land \neg r\} \).

Still using \( \alpha = (q \land \neg r) \lor (\neg q \land r) \), notice that \( B_{\text{rev}}^\alpha \neq B_{\text{bound}}^x \). But how different are \( B_{\text{rev}}^\alpha \) and \( B_{\text{bound}}^x \)? In the case of \( B_{\text{rev}}^\alpha \), we can compare it to \( B_{\text{bound}}^x \), \( B_{\text{rev}}^\alpha \). That is, \( B_{\text{rev}}^\alpha \) is defined as \( 0.5, 0.5, 0 \). That is, \( B_{\text{rev}}^\alpha \) and \( 0.5, 0.5, 0 \).

In the case of \( B_2 \), we could compare \( B_{\text{bound}}^x \).

It has been extensively argued [11, 26, 23] that maximum entropy is a reasonable inference mechanism, if not the most reasonable one (w.r.t. probability constraints). On the other hand, the boundary belief states method also seems like a very reasonable inference mechanism for revising BBs as defined here, in the sense that it causes information loss. Resolving this misalignment in the results of the two methods is an obvious task for future research.

An extended version of maximum entropy is minimum cross-entropy (MCE) [17, 5]:

**Definition 7 (Minimum Cross-Entropy)** The ‘directed divergence’ of distribution \( c \) from distribution \( b \) is defined as

\[
R(c, b) := \sum_{w \in W} c(w) \ln \frac{c(w)}{b(w)}.
\]

\( R(c, b) \) is undefined when \( b(w) = 0 \) while \( c(w) > 0 \); when \( c(w) = 0 \), \( R(c, b) = 0 \), because \( \lim_{x \to 0} \ln(x) = 0 \). Given new evidence
\( \phi \in L^\text{prob.} \), the distribution \( c \) satisfying \( \phi \) diverging least from current belief state \( b \) is

\[
\arg \min_{c \in \Pi, \phi \in \phi} R(c, b).
\]

**Definition 8 (MCI)** Then MCE inference (denoted (MCI)) is defined as

\[
b \text{ MCI } \alpha := \arg \min_{b' \in \Pi, R(b') \geq (\alpha)} R(b', b).
\]

In the following example, we interpret revision as MCE inference.

**Example 9** Suppose the vocabulary is \( \{ q, r \} \) and \( B_1 = \{ (q) \geq 0.6 \} \). Let \( \alpha \) be \( (q \land \neg r) \land (\neg q \land r) \). Then

\[
\Pi_{b_{\text{MCI}}}^{B_1} = \{ (11, 1.0), (10, 0.0), (01, 0.0), (00, 0.0) \},
\]

\[
(11, 0.0), (10, 1.0), (01, 0.0), (00, 0.0),
\]

\[
(11, 0.6), (10, 0.0), (01, 0.4), (00, 0.0),
\]

\[
(11, 0.6), (10, 0.0), (01, 0.0), (00, 0.4),
\]

\[
(11, 0.0), (10, 0.6), (01, 0.4), (00, 0.0),
\]

\[
(11, 0.0), (10, 0.6), (01, 0.0), (00, 0.4),
\]

\[
(\Pi_{b_{\text{MCI}}}^{B_1})_{B_2} = \{ (\phi_{11}) \leq 0, (\phi_{00}) \leq 0 \}.
\]

Note that if we let \( B' = \{ (q \land \neg r) \lor (\neg q \land r) \} = 1 \), then

\[
\Pi_{b_{\text{MCI}}}^{B_1} = \Pi_{b_{\text{MCI}}}^{B_2}.
\]

Recall from Example 6 that \( B' \) included \( (q \land \neg r) \geq 0.3 \). Hence, in this particular case, combining the boundary belief states approach with MCI results in a less informative revised belief base than when GI is used. The reason for the loss of information might be due to \( R(\cdot, \{ (11, 1.0), (10, 0.0), (00, 1.0), (00, 0.0) \}) \) and \( R(\cdot, \{ (11, 0.6), (10, 0.0), (01, 0.0), (00, 0.4) \}) \) being undefined: Recall that \( R(c, b) \) is undefined when \( c(w) = 0 \) while \( c(w) > 0 \). But then there is no belief state \( c \) for which \( c \models \alpha \) and \( R(\cdot, \cdot) \) is defined (with these two belief states as arguments). Hence, there are no revised counterparts of these two belief states in \( (\Pi_{b_{\text{MCI}}}^{B_1})_{\Pi_{b_{\text{MCI}}}^{B_2}} \). We would like to analyse MCI framework more within this approach. In the future, in the future, we would like to determine whether a statement like Theorem 1 holds for MCI too.

In MCE inference, \( b \)-consistency of evidence \( \phi \) is defined as:

There is a belief state \( e \) such that \( e \models \phi \) and \( e \) is totally continuous with respect to \( b \) (i.e., \( c(w) = 0 \) implies \( c(w) = 0 \)). MCE is undefined when the evidence is not \( b \)-consistent. This is analogous to Bayesian conditioning being undefined for \( b(\alpha) = 0 \). Obviously, this is a limitation of MCE because some belief states may not be considered as candidate revised belief states.

### 6 RATIONALITY POSTULATES

In this section, we assess the operation of revising a belief base \( B \) by \( a \) via GI (denoted \( B^a \)) with respect to several rationality postulates. Katsumo and Mendelzon [12] modified the eight AGM belief revision postulates [1] to the following six ((R1)-(R6)). The postulates are intended to be an ideal standard of rationality with respect to the behavior of any revision operator. We shall simply translate (\( \Rightarrow \)) each postulate into one appropriate for belief bases (in the notation of this paper) in order to get a sense of which ones are satisfied and which not. We then intend to use this information to guide us in drawing up appropriate generalised postulates in the future. In their notation [12], given a knowledge base represented by a propositional sentence \( \psi \) and an observation represented by a propositional sentence \( \mu \), \( \psi \circ \mu \) denotes the revision of \( \psi \) by \( \mu \).

(R1) \( \psi \circ \mu \) implies \( \mu \implies \mu \)

(RB1) \( B^a \models (\alpha) \).

**Proposition 1** Postulate (RB1) holds.

**Proof:**

By definition of GI, all probability mass is shifted to closest \( \alpha \)-worlds. For every revised boundary belief state \( b_{\text{bound}} \), it is thus the case that \( b_{\text{bound}}(\alpha) = 1 \). By the process of inducing \( B^a \), it must be that \( B^a \) entails \( (\alpha) = 1 \).

(R2) If \( \psi \land \mu \) is satisfiable, then \( \psi \circ \mu \equiv \psi \land \mu \implies (RB2) \) if \( B \cup \{ (\alpha) = 1 \} \) is satisfiable, then \( B^a \equiv B \cup \{ (\alpha) = 1 \} \).

**Proposition 2** Postulate (RB2) does not hold.

**Proof:**

Recall that \( B_1 = \{ (q) \geq 0.6 \} \). Example 6 shows that \( B^{(q \land \neg r) \lor (\neg q \land r)} \equiv B' = \{ (q \land \neg r) \lor (\neg q \land r) \} = 1 \), \( (q \land \neg r) \geq 0.3 \). Therefore, \( B' \equiv (q \land \neg r) \geq 0.3 \). But \( B_1 \cup \{ (q \land \neg r) \lor (\neg q \land r) \} = 1 \), although satisfiable, does not entail \( (q \land \neg r) \geq 0.3 \). Hence, \( B' \neq B \cup \{ (q \land \neg r) \lor (\neg q \land r) \} \).

Let \( B_1 = \{ (q \land r) = 0.2 \}, (q \land \neg r) = 0.8 \). Then \( B^a \equiv B \cup \{ (q) = 1 \} \), notice that \( B_1 \) specifies a particular belief state. One can see that Postulate (RB2) will hold whenever the \( B \) to be revised specifies a particular belief state (i.e., whenever \( |B'| = 1 \)).

(R3) If \( \psi \) is satisfiable, then \( \psi \circ \mu \) is also satisfiable \( \Rightarrow (RB3) \) if \( (\alpha) = 1 \) is satisfiable, then \( B^a \) is also satisfiable.

**Proposition 3** Postulate (RB3) holds.

**Proof:**

\( (\alpha) = 1 \) is satisfiable if \( \alpha \) is. And if \( \alpha \) is satisfiable, then every boundary belief state of \( B \) revised by \( \alpha \) is defined. Hence, \( B^a \) must be defined, i.e., satisfiable.

(R4) If \( \psi \equiv \psi' \land \mu \equiv \mu' \), then \( \psi \circ \mu \equiv \psi' \circ \mu' \Rightarrow (RB4) \) if \( B \equiv C \) and \( \alpha \equiv \beta \), then \( B^a \equiv C^a \).

**Proposition 4** Postulate (RB4) holds.

**Proof:**

By definition, if \( B \equiv C \), then \( \Pi^B = \Pi^C \). Hence, the boundary belief states for \( B \) and \( C \) are the same. And given \( \alpha \equiv \beta \), it must be the case that \( B^a \equiv C^a \).

(R5) \( \psi \circ \mu \land \phi \models \psi \circ \mu \land \phi \)

**Proposition 5** Postulate (RB5) does not hold.

**Proof:**

Let \( \alpha \models T \). Then \( B^a \equiv B \) and \( B^a \land \beta \equiv B^a \). Now let \( \beta \models (q \land \neg r) \lor (\neg q \land r) \). We can thus ask whether \( B \cup \{ (\beta) = 1 \} \models B^a \).

Consider Example 6. Recall that \( B^{(q \land \neg r) \lor (\neg q \land r)} \models (q \land \neg r) \geq 0.3 \). That \( q \in B \).

\[ B^{(q \land \neg r)} \models (q \land \neg r) \geq 0.3 \text{. And recall} \]

\[ \text{In these postulates, it is sometimes necessary to write an observation \( \alpha \)} \text{ as a BB, i.e., as \( \{ (\alpha) = 1 \} \) in the present framework, observations are regarded as certain.} \]

\[ \circ \text{ is some revision operator.} \]
that \( B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1 \) \( \not\subseteq \) \( q \land \neg r \geq 0.3 \). That is, \( \exists b \in B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} \) s.t. \( b \not\subseteq q \land \neg r \geq 0.3 \). Hence, \( B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1 \) \( \not\subseteq \) \( B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} \) and by the definition of \( \models \), \( B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1 \) \( \not\subseteq \) \( B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} \).

If \( B^\alpha \cup \{(\beta) = 1\} \) is not satisfiable, then (RB5) holds trivially. If \( B^\alpha \cup \{(\beta) = 1\} \) is satisfiable, then \( \alpha \) logically entails \( \beta \), implying that \( \alpha \land \beta \equiv \alpha \). Then we can ask whether \( B^\alpha \cup \{(\beta) = 1\} \) \( \models \) \( B^\alpha \)? Assuming \( B \) specifies a particular belief state, clearly \( B^\alpha \cup \{(\beta) = 1\} \) \( \models \) \( B^\alpha \) and (RB5) holds. So the difficulties come in when \( |\Pi^B| > 1 \).

(R6) If \( \psi \land \phi \models \psi \land \phi \), then \( B^\alpha \cup \{(\beta) = 1\} \) is satisfiable, then \( B^\alpha \land \beta \models B^\alpha \cup \{(\beta) = 1\} \).

**Proposition 6 Postulate (RB6) does not hold.**

**Proof:**

Let \( \alpha \subseteq x \). Then (RB6) becomes: If \( B \cup \{(\beta) = 1\} \) is satisfiable, then \( B^\alpha \models B \cup \{(\beta) = 1\} \).

Consider Example 6. Let \( \beta \models (q \land \neg r) \lor (\neg q \land r) \). Note that \( B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1 \) is satisfiable. Then by (RB6), \( B^\alpha \cup \{(q \land \neg r) \lor (\neg q \land r)\} \models B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1 \). But this is false:

Let \( b = \{0, 0.5, 0.5, 0.5\} \). Then \( \beta \models (q \land \neg r) \lor (\neg q \land r) \).

\begin{align*}
\beta \models B_1 & \models B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1, (q \land \neg r) \geq 0.3, \\
\beta \not\models B_1 & \models B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1, (q \land \neg r) \geq 0.3.
\end{align*}

Hence, \( \beta \models B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} \models B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1, (q \land \neg r) \geq 0.3 \).

Therefore, \( B^\alpha \cup \{(q \land \neg r) \lor (\neg q \land r)\} \models B_1 \cup \{(q \land \neg r) \lor (\neg q \land r)\} = 1, (q \land \neg r) \geq 0.3 \).

From the discussion of Postulate (RB5), we notice that given the antecedent of (RB6), \( B^\alpha \cup \{(\beta) = 1\} \) \( \models \) \( B^\alpha \). Thus, the consequent of (RB6) becomes \( \models B^\alpha \models B^\alpha \). This means that (RB6) holds when \( |\Pi^B| = 1 \). Again, the difficulties come in when \( |\Pi^B| > 1 \).

### 7 RELATED WORK

Voorbraak [27] proposed the partial probability theory (PTT), which allows probability assignments to be partially determined, and where there is a distinction between probabilistic information based on (i) hard background evidence and (ii) some assumptions. He does not explicitly define the "constraint language", however, from his examples and discussions, one can infer that he has something like the language \( L_{PTT} \) in mind: it contains all formulae which can be formed with sentences in our \( L_{prob} \) in combination with connectives \( \neg, \land \) and \( \lor \). A "belief state" in PTT is defined as the quadruple \( \langle \Omega, B, A, C \rangle \), where \( \Omega \) is a sample space, \( B \subset L_{PTT} \) is a set of probability constraints, \( A \subset L_{PTT} \) is a set of assumptions and \( C \subset W \) "represents specific information concerning the case at hand" (an observation or evidence).8 Our epistemic state can be expressed as a restricted PTT "belief state" by letting \( \Omega = W, B = B, A = \emptyset \) and \( C = \{w \in W : w \models \alpha\} \), where \( B \) is a belief base and \( \alpha \) is an observation in our notation.

Voorbraak [27] mentions that he will only consider conditioning where the evidence does not contradict the current beliefs. He defines the set of belief states corresponding to the conditionalized PPT “belief state” as \( \{b(\cdot | C) \in \Pi | b \in \Pi^{B\cup A}, b(C) > 0\} \). In our notation, this corresponds to \( \{b(\cdot | C)b) \in \Pi | b \in \Pi^B, b(\alpha), b(\alpha) > 0\} \), where \( \alpha \) corresponds to \( C \). Voorbraak [27] proposes constraining as an alternative to conditioning: Let \( \phi \in L_{prob} \) be a probability constraint.

In our notation, constraining \( \Pi^B \) on \( \phi \) produces \( \Pi^{B\cup \{\alpha\}} \).

Note that expanding a belief set reduces the number of models (worlds) and expanding a PPT "belief state" with extra constraints also reduces the number of models (belief states / probability distributions).

In the context of belief sets, it is possible to obtain any belief state from the ignorant belief state by a series of expansions. In PPT, constraining, but not conditioning, has the analogous property. This is one of the main reasons we prefer to constrain and not conditioning to be the probabilistic version of expansion. [27, p. 4]

But Voorbraak does not address the issue that \( C \) and \( \phi \) are different kinds of observations, so constraining, as defined here, cannot be an alternative to conditioning. \( C \) cannot be used directly for constraining and \( \phi \) cannot be used directly for conditioning.

W.l.o.g., we can assume \( C \) is represented by \( \alpha \). If we take \( b(\cdot | A) \) to be an expansion of \( \alpha \) whenever \( b(\alpha) > 0 \), then one might ask, Is it possible to obtain any belief base \( B' \) from the ignorant belief base \( B = \emptyset \) by a series of expansions, using our approach? The answer is, No. For instance, there is no observation or series of observations which can change \( B = \emptyset \) into \( B' = \{q, 0 \geq 0.6\} \). But if we were to allow sentences (constraints) in \( L_{prob} \) to be observations, then we could obtain any \( B' \) from the ignorant \( B \).

Groeneveld and Halpern [9] investigate what "update" (incorporation of an observation with current beliefs, such that the observation does not contradict the beliefs) means in a framework where beliefs are represented by a set of belief states. They state that the main purpose of their paper is to illustrate how different the set-of-distributions framework can be, "technically", from the standard single-distribution framework. They propose six postulates characterizing what properties an update function should have. They say that some of the postulates are obvious, some are highly improbable, and one probably too strong. Out of seven (families) of update functions only the one based on conditioning \( (\Pi_{\text{cond}}(\cdot)) \) and the one based on constraining \( (\Pi_{\text{cond}}(\cdot)) \) satisfy all six postulates, where \( \Pi_{\text{cond}}(\cdot) : \Pi \times b(\alpha) \in \Pi | b \in \Pi^B, b(\alpha) > 0 \) and where they interpret Voorbraak’s constraining [27] as \( \Pi_{\text{cond}}(\cdot) : \Pi \times b(\alpha) : b(\alpha) = 1 \). Groeneveld and Halpern [9] do not investigate the case when an observation must be incorporated while it is (possibly) inconsistent with the old beliefs (i.e., revision). It would be interesting to analyse the present work against their six postulates.

Kern-Isberner [14] develops a new perspective of probabilistic belief change. Based on the ideas of Alchourrón et al. [1] and Katsuno and Mendelzon [12] (KM), the operations of revision and update, respectively, are investigated within a probabilistic framework. She employs as basic knowledge structure as belief base (b, R), where b is a probability distribution (belief state) of background knowledge and R is a set of probabilistic conditionals of the form \( A \sim B[x] \) meaning the probability of \( B \) given \( A \) is \( x \). A universal inference operation basis on the techniques of optimum entropy — is introduced as an "adequate and powerful method to realize probabilistic belief change".

By having a belief state available in the belief base, minimum
cross-entropy can be used. The intention is then that an agent with belief base \((b, T)\) should always reason w.r.t. belief state \(b^T := \arg \min_{b \in \Pi_{\phi \in T}} R(c, b)\). Kern-Isberner [14] then defines the probabilistic belief revision of \((b, R)\) by evidence \(S\) as \((b, R \cup S)\). And the probabilistic belief update of \((b, R)\) by evidence \(S\) is defined as \((b^R, S)\).\(^9\) She distinguishes between revision as a knowledge adding process, and updating as a change-recording process. Kern-Isberner [14] sets up comparisons of maximum cross-entropy belief change process, and updating as a change-recording process. Kern-Isberner's vision, is inconsistent with the prior distribution \(b\), or, for revision, is inconsistent with \(b\) or the context \(R\), are not dealt with [14, p. 399, 400].

Having a belief state available for modification when new evidence is to be adopted is quite convenient. As Voorbraak [27] argues, however, an agent’s ignorance can hardly be represented in an epistemic state where a single belief state must always be chosen.

We would like to investigate the representation of conditional probabilistic information such as is done in the work of Kern-Isberner [14, 15] and Yue and Liu [29], for instance.

Yue and Liu [29] propose a probabilistic revision operation for imprecise probabilistic beliefs in the framework of Probabilistic Logic Programming (PLP). New evidence may be a probabilistic (conditional) formula and needs not be consistent with the original beliefs. Revision via imaging (e.g., GI) also overcomes this consistency issue. Essentially, their probabilistic epistemic states \(\Psi\) are induced from a PLP program which is a set of formulae, each formula having the form \((\psi \mid o)[l, u]\), meaning that the probability of the conditional \((\psi \mid o)\) lies in the interval \([l, u]\). The operator they propose has the characteristic that if an epistemic state \(\Psi\) represents a single probability distribution, revising collapses to Jeffrey's rule and Bayesian conditioning. They mention that it is required that the models (distributions) of \(\Psi\) is a convex set. There might thus be an opportunity to employ their revision operation on a representative set of boundary distributions as proposed in this paper.

Another PLP, proposed by Michels et al. [21], also allow for incomplete specification of probabilities. The language of their logic is however more expressive than ours, and they focus on inference (probabilistic query answering). They mention that they want to allow for learning in their system—one could possibly interpret some kinds of learning in this setting as belief revision.

\[8 \text{ CONCLUSION AND FUTURE DIRECTIONS} \]

In this paper, we propose an approach how to generate a new probabilistic belief base from an old one, given a new piece of non-probabilistic information, where a belief base is a finite set of sentences, each sentence stating the likelihood of a proposition about the world. In this framework, an agent’s belief base represents the set of belief states compatible with the sentences in it. In this sense, the agent is able to represent its knowledge and ignorance about the true state of the world.

We used a version of the so-called imaging approach to implement the revision operation.

Three methods were proposed: revising a finite set of ‘boundary belief states’ via generalized imaging, revising a finite set of ‘boundary belief states’ via minimum cross-entropy and revising a least biased belief state. We focussed on the first method and showed that the latter two give different results.

There were two main contribution of this paper. The first was to prove that the set of belief states satisfying \(B_{new}\) is exactly those belief states satisfying the original belief base, revised. The second was to uncover an interesting conflict in the results of the three belief base revision methods. It is worth further understanding the reasons behind such a difference, as such an investigation could give more insight about the mechanisms behind the methods and indicate possible pros and cons of each. Importantly, further analysis with respect to rationality postulated is necessary, as mentioned in § ??.

Such an analysis may also bring insights into the differing results.

The computational complexity of Min\((\cdot)\) is in \(O(|W|^2)\) and the complexity of GI is thus in \(O(|W|^2|W|^2)\) in the worst case. However, this complexity is highly dependent on the observation and the distance function. Note that \(|\Pi_{\phi \in T}^B| \leq |W|\), GI is applied to every belief state in \(\Pi_{\phi \in T}^B\). Hence, the complexity of the method, in the worst case, is in \(O(|W|)|W|^2\).

As far as we know, there is no analytic solution to determine the distribution in \(\Pi^B\) with maximum entropy / minimum cross-entropy. To narrow in on the computational complexity of these methods, we would have to know what class of optimization problem they are (convex?), and thus what techniques are used to solve them. Our knowledge in this area is lacking and it would require some time for investigation.

The proposal or design, and justification of rationality postulates similar to (RB1)-(RB6) in the section above is called for. An analysis of the postulates with respect to the revision operation must then be carried out. An attempt may be made to design the revision operation so as to make more of the postulates hold, or we may attempt to justify why our revision process does not / need not satisfy postulates it fails at.

Given that we have found that the belief base resulting from revising via the boundary-belief-states approach differs from the belief base resulting from revising via the representative-belief-state approach, the question arises, is it appropriate to use a representative belief state defined as the most entropic belief state of a given set \(\Pi^{14}\)? This is an important question, especially due to the popularity of employing the Maximum Entropy principle in cases of unspecified probabilistic knowledge [11, 8, 10, 27, 14, 16] and the principle’s well-behavedness [26, 22, 13].

As far as we know, imaging for belief change has never been applied to (conditional) probabilistic evidence. Due to issues with many revision methods required to be consistent with prior beliefs, and imaging not having this limitation, it might be worthwhile investigating.

The translation from the set of belief states back to a belief base is a mapping from every belief state to a probability formula. The size of the belief base is thus in the order of \(|W|^{perm}\), where \(|W|\) is already exponential in the size of \(P\), the set of atoms. As we saw in several examples in this paper, the new belief base often has a more concise equivalent counterpart. It would be useful to find a way to consistently determine more concise belief bases than our present approach does.

One of this paper’s reviewers mentioned credal sets, closed convex sets of probability distributions (which we had not heard of before). Another reviewer asked whether a similar result as Theorem 1 holds for Bayesian conditioning. This is an important question we would like to answer and which credal set theory [19, 28] may answer.

\(^9\) This is a very simplified version of what she presents. Please refer to the paper for details.
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